

COMMUTATORS OF AUTOMORPHIC COMPOSITION OPERATORS WITH ADJOINTS

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ABSTRACT. In this paper, we investigate the compactness of the commutator $[C_\psi^*, C_\varphi]$ on the Hardy space $H^2(B_N)$ or the weighted Bergman space $A_s^2(B_N)$ ($s > -1$), when φ and ψ are automorphisms of the unit ball B_N . We obtain that $[C_\psi^*, C_\varphi]$ is compact if and only if φ and ψ commute and they are both unitary. This generalizes the corresponding result in one variable. Moreover, our technique is different and simpler. In addition, we also discuss the commutator $[C_\psi^*, C_\varphi]$ on the Dirichlet space $\mathcal{D}(B_N)$, where φ and ψ are linear fractional self-maps or both automorphisms of B_N .

1. INTRODUCTION

Let B_N denote the unit ball of \mathbb{C}^N and let φ be a holomorphic self-map of B_N . We define the composition operator C_φ by $C_\varphi(f) = f \circ \varphi$, where f is analytic in B_N .

In this paper, we are interested in characterizing the compactness of the commutator $[C_\psi^*, C_\varphi] = C_\psi^* C_\varphi - C_\varphi C_\psi^*$ on some classical function spaces, when φ and ψ are automorphisms of B_N . The motivation comes from a recent work of Clifford et al. [4]. They considered when the commutator $[C_\psi^*, C_\varphi]$ is non-trivially compact on the Hardy space $H^2(D)$ for linear fractional self-maps φ and ψ of the unit disk D . Here, non-trivially compact means that $[C_\psi^*, C_\varphi]$ is compact but nonzero, moreover, both $C_\psi^* C_\varphi$ and $C_\varphi C_\psi^*$ are not compact. In particular, when φ and ψ are automorphisms of D , they showed that $[C_\psi^*, C_\varphi]$ is non-trivially compact if and only if both maps are rotations. All results were extended by MacCluer et al. [15] to the weighted Bergman space $A_s^2(D)$ ($s > -1$).

In Section 3, we first investigate the commutator $[C_\psi^*, C_\varphi]$ on the Hardy space $H^2(B_N)$ and the weighted Bergman space $A_s^2(B_N)$ ($s > -1$), where φ and ψ are automorphisms of B_N , neither φ nor ψ is the identity. In this case, we will prove that $[C_\psi^*, C_\varphi]$ is compact if and only if φ and ψ commute and both maps are unitary. This generalizes the result of automorphisms case in the unit disk to the unit ball. In order to deduce that φ and ψ commute when $[C_\psi^*, C_\varphi]$ is compact on $A_s^2(D)$, MacCluer et al. [15] had done lots of complicated calculations. It is difficult for us to use this similar idea. So in Section 3, we will find another simpler technique to solve similar problem in higher dimensions, which involved in the application of the semi-multiplication property for Toeplitz operators.

Furthermore, we will extend to discuss the compactness of $[C_\psi^*, C_\varphi]$ on the Dirichlet space $\mathcal{D}(B_N)$ in Section 4. Based on the adjoint formula for C_ψ on $\mathcal{D}(B_N)$, we obtain the necessary and sufficient condition for $[C_\psi^*, C_\varphi]$ to be compact in terms of linear fractional self-maps φ and ψ of B_N . As an immediate result, for automorphisms φ and

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ψ of B_N , we find that $[C_\psi^*, C_\varphi]$ is compact on $\mathcal{D}(B_N)$ if and only if φ and ψ commute. More specially, when φ and ψ are linear fractional self-maps of D , the condition for $[C_\psi^*, C_\varphi]$ to be compact on the Dirichlet space $\mathcal{D}(D)$ is the same as that on the Hardy space $H^2(D)$ and the weighted Bergman space $A_s^2(D)$ ($s > -1$).

2. PRELIMINARIES

Here we collect some necessary background information.

2.1. Analytic function spaces. Let ∂B_N denote the boundary of the unit ball B_N . The Hardy space $H^2(B_N)$ is defined by

$$H^2(B_N) = \{f \text{ analytic in } B_N : \|f\|^2 \equiv \sup_{0 < r < 1} \int_{\partial B_N} |f(r\zeta)|^2 d\sigma(\zeta) < \infty\},$$

where $d\sigma$ denotes the normalized surface measure on ∂B_N . The weighted Bergman space $A_s^2(B_N)$, for $s > -1$, is defined by

$$A_s^2(B_N) = \{f \text{ analytic in } B_N : \|f\|_s^2 \equiv \int_{B_N} |f(z)|^2 dv_s(z) < \infty\},$$

where $dv_s(z) = \frac{\Gamma(N+s+1)}{N!\Gamma(s+1)}(1-|z|^2)^s dv(z)$ and dv denotes the normalized volume measure on B_N . In this paper, we will often use \mathcal{H} to denote the Hardy space $H^2(B_N)$ or the weighted Bergman space $A_s^2(B_N)$. It is well known that both the Hardy space and the weighted Bergman space are the reproducing kernel Hilbert spaces, where the reproducing kernel is given by $K_w(z) = (1 - \langle z, w \rangle)^{-t}$ for $z, w \in B_N$, with $t = N$ for $H^2(B_N)$ and $t = N + s + 1$ for $A_s^2(B_N)$. So the normalized reproducing kernel is given by

$$k_w(z) = \frac{K_w(z)}{\|K_w\|_{\mathcal{H}}} = \frac{(1 - |w|^2)^{t/2}}{(1 - \langle z, w \rangle)^t}.$$

A multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N -tuple of non-negative integers α_i . The total order of a multi-index is given by $|\alpha| = |\alpha_1| + \dots + |\alpha_N|$. Let $\alpha! = \alpha_1! \cdots \alpha_N!$ and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ for $z = (z_1, \dots, z_N) \in \mathbb{C}^N$. An analytic function f in B_N has a power series representation

$$f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha},$$

where the sum is over all multi-indexes.

The Dirichlet space $\mathcal{D}(B_N)$ is defined as

$$\mathcal{D}(B_N) = \{f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha} \text{ analytic in } B_N : \|f\|_{\mathcal{D}^0}^2 \equiv \sum_{\alpha} |c_{\alpha}|^2 |\alpha| \frac{\alpha!}{|\alpha|!} < \infty\},$$

where the quantity $\|\cdot\|_{\mathcal{D}^0}$ defines a semi-norm on $\mathcal{D}(B_N)$. We equip it with the norm

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \|f\|_{\mathcal{D}^0}^2$$

and the inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$. So the reproducing kernel for $\mathcal{D}(B_N)$ is given by

$$K_w(z) = 1 + \log \frac{1}{1 - \langle z, w \rangle}, \quad z, w \in B_N.$$

When acting on the reproducing kernel, composition operators and Toeplitz operators have the following adjoint property:

$$C_{\varphi}^* T_h^* K_w = \overline{h(w)} K_{\varphi(w)}, \quad w \in B_N$$

for all analytic self-maps φ of B_N and $h \in L^\infty(B_N)$.

2.2. Adjoint formula. Given a bounded measurable complex-valued function b on ∂B_N (or B_N), the Toeplitz operator T_b on $H^2(B_N)$ (or $A_s^2(B_N)$) is defined by

$$T_b f = P(bf),$$

where P is the orthogonal projection from $L^2(\partial B_N)$ (or $L^2(B_N, dv_s)$) onto $H^2(B_N)$ (or $A_s^2(B_N)$). If b is analytic, then T_b is a multiplication by b .

In this paper, we will often use the semi-multiplication property for Toeplitz operators mod \mathcal{K} , where \mathcal{K} denotes the ideal of compact operators. That is, if $b \in L^\infty(\partial B_N)$ (or $L^2(B_N)$) and $h \in C(\overline{B_N})$, then $T_b T_h - T_{bh}$ is compact on $H^2(B_N)$ (or $A_s^2(B_N)$). This result on the Hardy space $H^2(B_N)$ is Proposition 1.4 in [17]; the Bergman space version comes from the proof of Theorem 1 in [5]; similar to the proof of Theorem 1 in [5], this property also can be extended to the weighted Bergman space $A_s^2(B_N)$ ($s > -1$). In the unit disk D , this fact has been well described in page 73 of [15].

Theorem A. *Suppose that*

$$\varphi(z) = \frac{az + b}{cz + d}$$

is a linear fractional self-map of D and $ad - bc \neq 0$. Then the adjoint of C_φ acting on the Hardy space $H^2(D)$ or the weighted Bergman space $A_s^2(D)$ ($s > -1$) is given by

$$C_\varphi^* = T_g C_\sigma T_h^*,$$

where

$$\sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}$$

is the Krein adjoint of φ , $g(z) = (-\bar{b}z + \bar{d})^{-t}$ and $h(z) = (cz + d)^t$ are in H^∞ with $t = 1$ for $H^2(D)$ and $t = s + 2$ for $A_s^2(D)$.

This adjoint formula of C_φ was first established on $H^2(D)$ by Cowen [6] and was generalized to $A_s^2(D)$ ($s > -1$) by Hurst [12]. In the unit ball B_N , Cowen and MacCluer [8] obtained the following similar adjoint formula for C_φ when acting on $H^2(B_N)$ or $A_s^2(B_N)$ ($s > -1$).

Theorem B. *Let*

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + d}$$

be a linear fractional self-map of B_N , where A is an $N \times N$ matrix, B and C are $N \times 1$ matrices, and d is a scalar. Then on the space \mathcal{H} ,

$$C_\varphi^* = T_g C_\sigma T_h^*,$$

where

$$\sigma(z) = \frac{A^* z - C}{\langle z, -B \rangle + \bar{d}},$$

$g(z) = (\langle z, -B \rangle + \bar{d})^{-t}$ and $h(z) = (\langle z, C \rangle + d)^t$ with $t = N$ when $\mathcal{H} = H^2(B_N)$ and $t = N + s + 1$ when $\mathcal{H} = A_s^2(B_N)$ ($s > -1$).

We will refer to the functions g, h and σ as the auxiliary functions of φ when they connected by the equation $C_\varphi^* = T_g C_\sigma T_h^*$. We also need frequently use the property: if $\varphi \in \text{Aut}(B_N)$ (or $\text{Aut}(D)$) then $\sigma = \varphi^{-1}$, where $\text{Aut}(B_N)$ (or $\text{Aut}(D)$) denotes the set of automorphisms of B_N (or D).

Moreover, we have a different adjoint formula for C_φ on the Dirichlet space $\mathcal{D}(B_N)$ as the following.

Theorem C. (Theorem 7 in [18]) *Let φ be a linear fractional self-map of B_N and K_w be the reproducing kernel of $\mathcal{D}(B_N)$. Then for $f \in \mathcal{D}(B_N)$, we have*

$$C_\varphi^* f = f(0)K_{\varphi(0)} + C_\sigma f - f(\sigma(0)),$$

where σ is the Krein adjoint of φ .

This means that the adjoint of C_φ on $\mathcal{D}(B_N)$ can be identified as another composition operator and a rank 2 operator. For the unit disk case, this adjoint formula on the Dirichlet space $\mathcal{D}(D)$ was given by Gallardo-Gutiérrez and Montes-Rodríguez (see Theorem 3.3 in [10]).

It is well known that when φ is a linear fractional self-map of B_N , then C_φ is compact on the space mentioned in this paper if and only if $\|\varphi\|_\infty < 1$. Let φ be a linear fractional self-map of D with a fixed point $\omega \in \partial D$, if ω is the only fixed point for φ , we say that φ is parabolic. If φ has an additional fixed point, we call φ hyperbolic.

2.3. Julia-Carathéodory theorem. Given $\zeta \in \partial B_N$, a continuous function $\gamma : [0, 1] \rightarrow B_N$ with $\lim_{t \rightarrow 1} \gamma(t) = \zeta$ is said to a restricted ζ -curve if

$$\lim_{t \rightarrow 1} \frac{|\gamma(t) - \langle \gamma(t), \zeta \rangle|^2}{1 - |\langle \gamma(t), \zeta \rangle|^2} = 0 \quad \text{and} \quad \sup_{0 \leq t < 1} \frac{|\zeta - \langle \gamma(t), \zeta \rangle|}{1 - |\langle \gamma(t), \zeta \rangle|} < \infty.$$

If $\lim_{t \rightarrow 1} f(\gamma(t)) = f(\zeta)$ for every restricted ζ -curve γ , we say that $f : B_N \rightarrow \mathbb{C}$ has restricted limit and write $R \lim_{z \rightarrow \zeta} f(z) = f(\zeta)$.

Let φ be a holomorphic self-map of B_N . We say that φ has finite angular derivative at $\zeta \in \partial B_N$, if there exists a point $\eta \in \partial B_N$ so that

$$A_\varphi(\zeta) = R \lim_{z \rightarrow \zeta} \frac{1 - \langle \varphi(z), \eta \rangle}{1 - \langle z, \zeta \rangle}$$

exists. We write $\varphi_\eta := \langle \varphi, \eta \rangle$ and $D_\zeta = \frac{\partial}{\partial \zeta}$ for the directional derivative in the direction of ζ , and we put

$$d_\varphi(\zeta) = \liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}$$

The following is the Julia-Carathéodory theorem for the ball (see Theorem 8.5.6 in [19] or Theorem 2.2 in [3])

Theorem D. *Let φ be a holomorphic self-map of B_N and $\zeta \in \partial B_N$. The following statements are equivalent:*

- (1) φ has finite angular derivative at ζ .
- (2) $d_\varphi(\zeta) < \infty$.
- (3) φ has restricted limit $\eta \in \partial B_N$ at ζ and $D_\zeta \varphi_\eta(z) = \langle \varphi'(z)\zeta, \eta \rangle$ has finite restricted limit at ζ .

Moreover, when these conditions hold, the following statements hold:

- (4) $D_\zeta \varphi_\eta(z) = \langle \varphi'(z)\zeta, \eta \rangle$ has restricted limit at ζ with $D_\zeta \varphi_\eta(\zeta) = d_\varphi(\zeta)$.
- (5) $A_\varphi(\zeta) = d_\varphi(\zeta)$.
- (6) $\frac{\varphi_{\eta^\perp}(z)}{1 - \langle z, \zeta \rangle}$ has restricted limit 0 at ζ for any $\eta^\perp \in \partial B_N$ orthogonal to η .

3. THE COMMUTATOR ON THE HARDY SPACE AND THE WEIGHTED BERGMAN SPACE

In this section, we discuss the compactness of the commutator $[C_\psi^*, C_\varphi]$ on the Hardy space $H^2(B_N)$ and the weighted Bergman space $A_s^2(B_N)$ ($s > -1$), when φ and ψ are automorphisms of B_N . We will show the following main theorem.

Theorem 3.1. *Let φ and ψ be automorphisms of B_N , neither φ nor ψ is the identity. The commutator $[C_\psi^*, C_\varphi]$ is compact on $H^2(B_N)$ or $A_s^2(B_N)$ ($s > -1$) if and only if φ and ψ commute and both maps are unitary.*

Before we give a proof for Theorem 3.1, we need some useful lemmas. The first lemma is analogous to Lemma 3.2 of [4] and Lemma 4.3 of [15]. But the result has been improved. Recall that for $w \in B_N$, the function k_w is the normalized reproducing kernel given by $k_w = K_w / \|K_w\|_{\mathcal{H}}$.

Lemma 3.2. *Assume that φ and ψ are holomorphic self-maps of B_N . Suppose that there exist points ζ_1 and ζ_2 on ∂B_N such that $\varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial B_N$ and $A_\varphi(\zeta_1)$ and $A_\psi(\zeta_2)$ exist. Then on the space \mathcal{H} ,*

$$\lim_{r \rightarrow 1} \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle = \left(\frac{2}{d_\varphi(\zeta_1) + d_\psi(\zeta_2)} \right)^t > 0,$$

where $t = N$ when $\mathcal{H} = H^2(B_N)$ and $t = N + s + 1$ when $\mathcal{H} = A_s^2(B_N)$ ($s > -1$).

Proof. Let U be a unitary map of B_N so that $U\omega = e_1$. Then $U\varphi(\zeta_1) = U\psi(\zeta_2) = U\omega = e_1$. Write $\phi = U\varphi$ and $\rho = U\psi$, we see that

$$\langle C_\rho^* k_{r\zeta_2}, C_\phi^* k_{r\zeta_1} \rangle = \langle C_U^* C_\psi^* k_{r\zeta_2}, C_U^* C_\varphi^* k_{r\zeta_1} \rangle = \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle.$$

Thus we may assume $\omega = e_1$. Note that

$$\frac{1}{\langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle} = \frac{\|K_{r\zeta_1}\|_{\mathcal{H}} \|K_{r\zeta_2}\|_{\mathcal{H}}}{\langle K_{\psi(r\zeta_2)}, K_{\varphi(r\zeta_1)} \rangle} = \left(\frac{1 - \langle \varphi(r\zeta_1), \psi(r\zeta_2) \rangle}{1 - r^2} \right)^t$$

and

$$\begin{aligned} \frac{1 - \langle \varphi(r\zeta_1), \psi(r\zeta_2) \rangle}{1 - r^2} &= \frac{1 - |\psi(r\zeta_2)|^2}{1 - r^2} + \frac{|\psi(r\zeta_2)|^2 - \langle \varphi(r\zeta_1), \psi(r\zeta_2) \rangle}{1 - r^2} \\ &= \frac{1 - |\psi(r\zeta_2)|^2}{1 - r^2} + \frac{\langle \psi(r\zeta_2) - \varphi(r\zeta_1), \psi(r\zeta_2) \rangle}{1 - r^2}. \end{aligned}$$

Now, we calculate that

$$\begin{aligned} &\frac{\langle \psi(r\zeta_2) - \varphi(r\zeta_1), \psi(r\zeta_2) \rangle}{1 - r^2} \\ &= \frac{\psi_1(r\zeta_2) - \varphi_1(r\zeta_1)}{1 - r^2} \cdot \overline{\psi_1(r\zeta_2)} + \sum_{j=2}^N \frac{\psi_j(r\zeta_2) - \varphi_j(r\zeta_1)}{1 - r^2} \cdot \overline{\psi_j(r\zeta_2)} \\ &= \left(\frac{1 - \langle \varphi(r\zeta_1), e_1 \rangle}{1 - \langle r\zeta_1, \zeta_1 \rangle} - \frac{1 - \langle \psi(r\zeta_2), e_1 \rangle}{1 - \langle r\zeta_2, \zeta_2 \rangle} \right) \frac{\overline{\psi_1(r\zeta_2)}}{1 + r} \\ &\quad + \sum_{j=2}^N \left(\frac{\psi_j(r\zeta_2)}{1 - \langle r\zeta_2, \zeta_2 \rangle} - \frac{\varphi_j(r\zeta_1)}{1 - \langle r\zeta_1, \zeta_1 \rangle} \right) \frac{\overline{\psi_j(r\zeta_2)}}{1 + r}. \end{aligned}$$

Since φ and ψ have finite angular derivatives respectively at ζ_1 and ζ_2 , by Theorem D(5), we get that

$$\lim_{r \rightarrow 1} \frac{1 - \langle \varphi(r\zeta_1), e_1 \rangle}{1 - \langle r\zeta_1, \zeta_1 \rangle} = A_\varphi(\zeta_1) = d_\varphi(\zeta_1)$$

and

$$\lim_{r \rightarrow 1} \frac{1 - \langle \psi(r\zeta_2), e_1 \rangle}{1 - \langle r\zeta_2, \zeta_2 \rangle} = A_\psi(\zeta_2) = d_\psi(\zeta_2).$$

Moreover, by Theorem D(6),

$$\lim_{r \rightarrow 1} \frac{\varphi_j(r\zeta_2)}{1 - \langle r\zeta_1, \zeta_1 \rangle} = 0 \quad \text{and} \quad \lim_{r \rightarrow 1} \frac{\psi_j(r\zeta_2)}{1 - \langle r\zeta_2, \zeta_2 \rangle} = 0$$

hold for $2 \leq j \leq N$. Therefore,

$$\begin{aligned} \lim_{r \rightarrow 1} \frac{1}{\langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle} &= \lim_{r \rightarrow 1} \left(\frac{1 - \langle \varphi(r\zeta_1), \psi(r\zeta_2) \rangle}{1 - r^2} \right)^t \\ &= \lim_{r \rightarrow 1} \left(\frac{1 - |\psi(r\zeta_2)|^2}{1 - r^2} + \frac{\langle \psi(r\zeta_2) - \varphi(r\zeta_1), \psi(r\zeta_2) \rangle}{1 - r^2} \right)^t \\ &= \left[d_\psi(\zeta_2) + \frac{1}{2}(d_\varphi(\zeta_1) - d_\psi(\zeta_2)) \right]^t \\ &= \left(\frac{d_\varphi(\zeta_1) + d_\psi(\zeta_2)}{2} \right)^t. \end{aligned}$$

and we obtain the desire conclusion. \square

Lemma 3.3. *Assume that φ and ψ are holomorphic self-maps of D . Suppose that there exist points ζ_1 and ζ_2 on ∂D such that $\varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial D$ and the angular derivatives $\varphi'(\zeta_1)$ and $\psi'(\zeta_2)$ exist. Then on $H^2(D)$ or $A_s^2(D)$ ($s > -1$),*

$$\lim_{r \rightarrow 1} \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle = \left(\frac{2}{|\varphi'(\zeta_1)| + |\psi'(\zeta_2)|} \right)^t > 0,$$

where

$$k_w(z) = \left(\frac{\sqrt{1 - |w|^2}}{1 - z\bar{w}} \right)^t, \quad z, w \in D$$

is the normalized reproducing kernel with $t = 1$ for $H^2(D)$ and $t = s + 2$ for $A_s^2(D)$.

This is an easy corollary of Lemma 3.2. In fact, Lemma 3.3 can also be immediately obtained if we notice that $\zeta_1 \varphi'(\zeta_1) \omega = |\varphi'(\zeta_1)|$ and $\zeta_2 \psi'(\zeta_2) \omega = |\psi'(\zeta_2)|$ from the Julia-Carathéodory Theorem of the unit disk (see Theorem 2.44 of [7]). Because Lemma 3.2 of [4] and Lemma 4.3 of [15] have given that

$$\lim_{r \rightarrow 1} \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle = \left(\frac{2\omega}{2\omega|\psi'(\zeta_1)| + |\zeta_1 \varphi'(\zeta_1) - \zeta_2 \psi'(\zeta_2)|} \right)^t,$$

we only need multiple $\bar{\omega}$ on numerator and denominator of the above fraction to deduce the limit in Lemma 3.3.

We now show that when φ and ψ are automorphisms of D , in order for the commutator $[C_\psi^*, C_\varphi]$ to be compact, the inducing maps φ and ψ must commute. This result on the Hardy space $H^2(D)$ is Lemma 5.1 of [4], and it on the weighted Bergman space $A_s^2(D)$ is Theorem 4.6 of [15]. Their technique is similar, but on $A_s^2(D)$, some calculations involved are very complex. It is almost impossible to do similar calculations in higher dimensions. We have to find other method which can be used on $H^2(D)$ and $A_s^2(D)$ simultaneously and which can also be extended to the unit ball.

Lemma 3.4. *Assume that φ and ψ are automorphisms of D . If the commutator $[C_\psi^*, C_\varphi]$ is compact on $H^2(D)$ or $A_s^2(D)$ ($s > -1$), then φ and ψ commute.*

Proof. This is Lemma 5.1 in [4] and Theorem 4.6 in [15]. We will give another simpler proof and we only focus on the Hardy space $H^2(D)$. Set

$$\varphi(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad \psi(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

with normalization $a_1 d_1 - c_1 b_1 = 1$ and $a_2 d_2 - c_2 b_2 = 1$. By Theorem A, we have

$$C_\varphi = T_{h_1} C_{\sigma_1}^* T_{g_1}^* \quad \text{and} \quad C_\psi = T_{h_2} C_{\sigma_2}^* T_{g_2}^*,$$

where g_1, h_1, σ_1 and g_2, h_2, σ_2 are respectively the auxiliary functions for φ and ψ on Theorem A. Since φ and ψ are automorphisms of D , we have $\sigma_1 = \varphi^{-1}$ and $\sigma_2 = \psi^{-1}$.

For $w \in D$, let K_w denote the reproducing kernel and k_w be the normalized reproducing kernel respectively given by

$$K_w(z) = \frac{1}{1 - z\bar{w}} \quad \text{and} \quad k_w(z) = \frac{K_w(z)}{\|K_w\|} = \frac{\sqrt{1 - |w|^2}}{1 - z\bar{w}},$$

where $\|\cdot\|$ denotes the norm of $H^2(D)$. Now, using the formula $C_\phi^* T_b^* K_w = \overline{b(w)} K_{\phi(w)}$, we get that

$$\begin{aligned} \langle C_\varphi K_z, C_\psi K_w \rangle &= \langle T_{h_1} C_{\sigma_1}^* T_{g_1}^* K_z, T_{h_2} C_{\sigma_2}^* T_{g_2}^* K_w \rangle \\ &= g_1(z) \overline{g_2(w)} \langle T_{h_1} K_{\sigma_1(z)}, T_{h_2} K_{\sigma_2(w)} \rangle \\ &= g_1(z) \overline{g_2(w)} \langle K_{\sigma_1(z)}, T_{h_1}^* T_{h_2} K_{\sigma_2(w)} \rangle. \end{aligned}$$

Since $h_1(z) = c_1 z + d_1$ and $h_2(z) = c_2 z + d_2$ are in H^∞ , using the semi-multiplicative property for Toeplitz operators mod \mathcal{K} as mentioned in Section 2, we see

$$T_{h_1}^* T_{h_2} = T_{\overline{h_1}} T_{h_2} = T_{\overline{h_1} h_2} = T_{h_2} \overline{h_1} = T_{h_2} T_{\overline{h_1}} + L = T_{h_2} T_{h_1}^* + L,$$

where L is a compact operator on $H^2(D)$. It follows that

$$\begin{aligned} \langle C_\varphi K_z, C_\psi K_w \rangle &= g_1(z) \overline{g_2(w)} \langle K_{\sigma_1(z)}, T_{h_1}^* T_{h_2} K_{\sigma_2(w)} \rangle \\ &= g_1(z) \overline{g_2(w)} \langle K_{\sigma_1(z)}, T_{h_2} T_{h_1}^* K_{\sigma_2(w)} \rangle + g_1(z) \overline{g_2(w)} \langle K_{\sigma_1(z)}, L K_{\sigma_2(w)} \rangle \\ &= g_1(z) \overline{g_2(w)} \langle T_{h_2}^* K_{\sigma_1(z)}, T_{h_1}^* K_{\sigma_2(w)} \rangle + g_1(z) \overline{g_2(w)} \langle K_{\sigma_1(z)}, L K_{\sigma_2(w)} \rangle \\ &= g_1(z) \overline{g_2(w)} h_2 \circ \sigma_1(z) \overline{h_1 \circ \sigma_2(w)} \langle K_{\sigma_1(z)}, K_{\sigma_2(w)} \rangle \\ &\quad + g_1(z) \overline{g_2(w)} \langle K_{\sigma_1(z)}, L K_{\sigma_2(w)} \rangle. \end{aligned}$$

Fix $\omega \in \partial D$, since φ and ψ are automorphisms, there exist ζ_1 and ζ_2 on ∂D such that $\varphi(\zeta_1) = \psi(\zeta_2) = \omega$. Setting $z = r\zeta_2$ and $w = r\zeta_1$, then

$$\begin{aligned} \lim_{r \rightarrow 1^-} \langle C_\varphi k_{r\zeta_2}, C_\psi k_{r\zeta_1} \rangle &= \lim_{r \rightarrow 1^-} \langle C_\varphi \frac{K_{r\zeta_2}}{\|K_{r\zeta_2}\|}, C_\psi \frac{K_{r\zeta_1}}{\|K_{r\zeta_1}\|} \rangle \\ &= \lim_{r \rightarrow 1^-} (1 - r^2) \langle C_\varphi K_{r\zeta_2}, C_\psi K_{r\zeta_1} \rangle \\ &= \lim_{r \rightarrow 1^-} (1 - r^2) g_1(r\zeta_2) \overline{g_2(r\zeta_1)} h_2 \circ \sigma_1(r\zeta_2) \overline{h_1 \circ \sigma_2(r\zeta_1)} \langle K_{\sigma_1(r\zeta_2)}, K_{\sigma_2(r\zeta_1)} \rangle \\ &\quad + \lim_{r \rightarrow 1^-} (1 - r^2) g_1(r\zeta_2) \overline{g_2(r\zeta_1)} \langle K_{\sigma_1(r\zeta_2)}, L K_{\sigma_2(r\zeta_1)} \rangle \\ &=: I + II. \end{aligned}$$

Note that $\{k_w\}$ is a weakly convergent sequence as $|w| \rightarrow 1$ with $\|k_w\| = 1$ and L is compact, we see that

$$\lim_{|w| \rightarrow 1} \sqrt{1 - |w|^2} \|LK_w\| = \lim_{|w| \rightarrow 1} \|Lk_w\| = 0.$$

Which gives that

$$\begin{aligned} & \lim_{r \rightarrow 1^-} \sqrt{1 - r^2} \|LK_{\sigma_2(r\zeta_1)}\| \\ &= \lim_{r \rightarrow 1^-} \sqrt{1 - |\sigma_2(r\zeta_1)|^2} \|LK_{\sigma_2(r\zeta_1)}\| \cdot \sqrt{\frac{1 - r^2}{1 - |\sigma_2(r\zeta_1)|^2}} \\ &= 0 \cdot \frac{1}{|\sigma'_2(\zeta_1)|^{1/2}} = 0. \end{aligned}$$

Hence,

$$\begin{aligned} & \lim_{r \rightarrow 1^-} (1 - r^2) |g_1(r\zeta_2) \overline{g_2(r\zeta_1)}| < K_{\sigma_1(r\zeta_2)}, LK_{\sigma_2(r\zeta_1)} > | \\ & \leq \lim_{r \rightarrow 1^-} (1 - r^2) |g_1(r\zeta_2) \overline{g_2(r\zeta_1)}| \cdot \|K_{\sigma_1(r\zeta_2)}\| \cdot \|LK_{\sigma_2(r\zeta_1)}\| \\ &= \lim_{r \rightarrow 1^-} |g_1(r\zeta_2) \overline{g_2(r\zeta_1)}| \sqrt{\frac{1 - r^2}{1 - |\sigma_1(r\zeta_2)|^2}} \cdot \sqrt{1 - r^2} \|LK_{\sigma_2(r\zeta_1)}\| \\ &= |g_1(\zeta_2) \overline{g_2(\zeta_1)}| \frac{1}{|\sigma'_1(\zeta_2)|^{1/2}} \cdot 0 = 0 \end{aligned}$$

and so $II = 0$.

Now, using $\sigma_1 = \varphi^{-1}$ and $\sigma_2 = \psi^{-1}$, we calculate that

$$\begin{aligned} I &= \lim_{r \rightarrow 1^-} (1 - r^2) g_1(r\zeta_2) \overline{g_2(r\zeta_1)} h_2 \circ \sigma_1(r\zeta_2) \overline{h_1 \circ \sigma_2(r\zeta_1)} < K_{\sigma_1(r\zeta_2)}, K_{\sigma_2(r\zeta_1)} > \\ &= \lim_{r \rightarrow 1^-} \overline{h_1 \circ \psi^{-1}(r\zeta_1)} g_1(r\zeta_2) h_2 \circ \varphi^{-1}(r\zeta_2) \overline{g_2(r\zeta_1)} \frac{1 - r^2}{1 - \langle \psi^{-1}(r\zeta_1), \varphi^{-1}(r\zeta_2) \rangle}. \end{aligned}$$

It is easy to see that

$$\lim_{r \rightarrow 1^-} \frac{1 - r^2}{1 - \langle \psi^{-1}(r\zeta_1), \varphi^{-1}(r\zeta_2) \rangle} = 0$$

unless $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$. Next, we assume $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$. By Lemma 3.3, we get

$$\begin{aligned} \lim_{r \rightarrow 1} \frac{1 - r^2}{1 - \langle \psi^{-1}(r\zeta_1), \varphi^{-1}(r\zeta_2) \rangle} &= \lim_{r \rightarrow 1} \langle C_{\varphi^{-1}}^* k_{r\zeta_2}, C_{\psi^{-1}}^* k_{r\zeta_1} \rangle \\ &= \frac{2}{|(\psi^{-1})'(\zeta_1)| + |(\varphi^{-1})'(\zeta_2)|}. \end{aligned}$$

On the other hand, an easy computation shows that

$$h_1(z) \overline{g_1 \circ \varphi(z)} = \frac{c_1 z + d_1}{a_1 - c_1 \varphi(z)} = \frac{c_1 z + d_1}{a_1 - c_1 \frac{a_1 z + b_1}{c_1 z + d_1}} = \frac{|c_1 z + d_1|^2}{a_1 d_1 - c_1 b_1} = |c_1 z + d_1|^2$$

and similarly

$$h_2(z) \overline{g_2 \circ \psi(z)} = |c_2 z + d_2|^2.$$

This fact has been proved by MacCluer and Pons [16] for the the unit ball case. Combining this with $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$, we obtain

$$\begin{aligned} \overline{h_1 \circ \psi^{-1}(\zeta_1)} g_1(\zeta_2) &= \overline{h_1 \circ \varphi^{-1}(\zeta_2)} g_1 \circ \varphi \circ \varphi^{-1}(\zeta_2) \\ &= |c_1 \varphi^{-1}(\zeta_2) + d_1|^2 = \left| c_1 \frac{d_1 \zeta_2 - b_1}{-c_1 \zeta_2 + a_1} + d_1 \right|^2 \\ &= \frac{1}{|a_1 - c_1 \zeta_2|^2} = |(\varphi^{-1})'(\zeta_2)| \end{aligned}$$

and

$$\begin{aligned} h_2 \circ \varphi^{-1}(\zeta_2) \overline{g_2(\zeta_1)} &= h_2 \circ \psi^{-1}(\zeta_1) \overline{g_2 \circ \psi \circ \psi^{-1}(\zeta_1)} \\ &= |c_2 \psi^{-1}(\zeta_1) + d_2|^2 = |(\psi^{-1})'(\zeta_1)|. \end{aligned}$$

As a result, we show that

$$I = |(\varphi^{-1})'(\zeta_2)| |(\psi^{-1})'(\zeta_1)| \frac{2}{|(\psi^{-1})'(\zeta_1)| + |(\varphi^{-1})'(\zeta_2)|}$$

if $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$; Otherwise $I = 0$. Moreover, since $\varphi(\zeta_1) = \psi(\zeta_2) = \omega \in \partial D$, if $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$ then we have

$$\psi^{-1} \circ \varphi^{-1}(\omega) = \varphi^{-1} \circ \psi^{-1}(\omega).$$

This means φ^{-1} and ψ^{-1} commute and hence φ and ψ commute. Immediately, we get

$$(\psi^{-1})'(\varphi^{-1}(\omega))(\varphi^{-1})'(\omega) = (\varphi^{-1})'(\psi^{-1}(\omega))(\psi^{-1})'(\omega),$$

i.e.

$$(\psi^{-1})'(\zeta_1) \frac{1}{\varphi'(\zeta_1)} = (\varphi^{-1})'(\zeta_2) \frac{1}{\psi'(\zeta_2)}.$$

Thus, the above discussions deduce that

$$\begin{aligned} \lim_{r \rightarrow 1^-} \langle C_\varphi k_{r\zeta_2}, C_\psi k_{r\zeta_1} \rangle &=: I + II \\ &= |(\varphi^{-1})'(\zeta_2)| |(\psi^{-1})'(\zeta_1)| \frac{2}{|(\psi^{-1})'(\zeta_1)| + |(\varphi^{-1})'(\zeta_2)|} + 0 \\ &= \left| (\psi^{-1})'(\zeta_1) \frac{\psi'(\zeta_2)}{\varphi'(\zeta_1)} \right| |(\psi^{-1})'(\zeta_1)| \frac{2}{|(\psi^{-1})'(\zeta_1)| + \left| (\psi^{-1})'(\zeta_1) \frac{\psi'(\zeta_2)}{\varphi'(\zeta_1)} \right|} \\ &= |(\psi^{-1})'(\zeta_1) \psi'(\zeta_2)| \frac{2}{|\varphi'(\zeta_1)| + |\psi'(\zeta_2)|} \end{aligned}$$

under the condition of $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$. Otherwise, it is zero.

At last, by our hypothesis that $[C_\psi^*, C_\varphi]$ is compact on $H^2(D)$, we see that

$$\begin{aligned} 0 &= \lim_{r \rightarrow 1^-} \|[C_\psi^*, C_\varphi] k_{r\zeta_2}\| \geq \lim_{r \rightarrow 1^-} | \langle [C_\psi^*, C_\varphi] k_{r\zeta_2}, k_{r\zeta_1} \rangle | \\ &= \lim_{r \rightarrow 1^-} | \langle C_\varphi k_{r\zeta_2}, C_\psi k_{r\zeta_1} \rangle - \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle |. \end{aligned}$$

Using Lemma 3.3 again, we know that

$$\lim_{r \rightarrow 1^-} \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle = \frac{2}{|\varphi'(\zeta_1)| + |\psi'(\zeta_2)|}.$$

It follows that

$$\lim_{r \rightarrow 1^-} \langle C_\varphi k_{r\zeta_2}, C_\psi k_{r\zeta_1} \rangle = \lim_{r \rightarrow 1^-} \langle C_\psi^* k_{r\zeta_2}, C_\varphi^* k_{r\zeta_1} \rangle \neq 0.$$

Combining this with the previous conclusion, we must have $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$. Moreover, the above equality implies

$$|(\psi^{-1})'(\zeta_1)\psi'(\zeta_2)| = 1.$$

(We don't know how to use this conclusion, but it maybe have some independent interest.) Note that we have obtained that φ and ψ commute from $\varphi^{-1}(\zeta_2) = \psi^{-1}(\zeta_1)$. Therefore, $[C_\psi^*, C_\varphi]$ is compact must deduce that φ and ψ commute. \square

In the proof of Lemma 3.4, replacing Lemma 3.3 by Lemma 3.2, we can deduce the similar result for the unit ball. We state it as the following and omit its proof.

Lemma 3.5. *If $\varphi, \psi \in \text{Aut}(B_N)$ and $[C_\psi^*, C_\varphi]$ is compact on $H^2(B_N)$ or $A_s^2(B_N)$ ($s > -1$), then φ and ψ commute.*

Next, we will give a complete proof for our main theorem. That is, we will show that when φ and ψ are automorphisms of B_N , the compactness of $[C_\psi^*, C_\varphi]$ implies that both φ and ψ are unitary. In the proof of this similar conclusion, the technique treated on $A_s^2(D)$ (see [15]) is different from that on $H^2(D)$ (see [4]). On $A_s^2(D)$, the polar decomposition of C_φ was used. I think which also holds on the unit ball, so we can use similar idea as on $A_s^2(D)$ to complete the proof of this result. However, in order to exhibit the special property of composition operator C_φ when φ is an automorphism of B_N , we try to use the following interesting lemmas to prove this result.

Lemma 3.6. *Suppose that φ is an automorphism of B_N . Then on the space \mathcal{H} ,*

$$C_\varphi^* = T_f C_\varphi^{-1} = T_f C_{\varphi^{-1}},$$

where T_f is the Toeplitz operator with symbol

$$f(z) = \left(\frac{1 - |\varphi(0)|^2}{|1 - \langle z, \varphi(0) \rangle|^2} \right)^t$$

with $t = N$ when $\mathcal{H} = H^2(B_N)$ and $t = N + s + 1$ when $\mathcal{H} = A_s^2(B_N)$ ($s > -1$).

This result on the Hardy space $H^2(B_N)$ and the Bergman space $A^2(B_N)$ was established by Bourdon and MacCluer [2]. The extension to the weighted space $A_s^2(B_N)$ can be obtained similarly when using the change of variables formula in Proposition 1.13 of [20]. For the case of the unit disk, please see Theorem 4.2 of [15].

Lemma 3.7. *Let φ be an automorphism of B_N . If f is continuous on ∂B_N or $\overline{B_N}$, then*

$$C_\varphi T_f - T_{f \circ \varphi} C_\varphi$$

is compact when acting on $H^2(B_N)$ or $A_s^2(B_N)$ ($s > -1$).

Proof. Let φ be an automorphism of B_N and $a = \varphi^{-1}(0)$. By Theorem 2.2.5 of [19], the identity

$$1 - \langle \varphi(z), \zeta \rangle = 1 - \langle \varphi(z), \varphi \circ \varphi^{-1}(\zeta) \rangle = \frac{(1 - |a|^2)(1 - \langle z, \varphi^{-1}(\zeta) \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, \varphi^{-1}(\zeta) \rangle)}$$

holds for all $z \in \overline{B_N}$ and $\zeta \in \partial B_N$. Now, using this identity, for any $g \in H^2(B_N)$, we get that

$$\begin{aligned}
 (C_\varphi T_f g)(z) &= (T_f g)(\varphi(z)) = \int_{\partial B_N} f(\zeta) g(\zeta) \frac{1}{(1 - \langle \varphi(z), \zeta \rangle)^N} d\sigma(\zeta) \\
 &= \int_{\partial B_N} f(\zeta) g(\zeta) \frac{(1 - \langle z, a \rangle)^N (1 - \langle a, \varphi^{-1}(\zeta) \rangle)^N}{(1 - |a|^2)^N (1 - \langle z, \varphi^{-1}(\zeta) \rangle)^N} d\sigma(\zeta) \\
 &= \int_{\partial B_N} f \circ \varphi(\eta) g \circ \varphi(\eta) \frac{(1 - \langle z, a \rangle)^N (1 - \langle a, \eta \rangle)^N}{(1 - |a|^2)^N (1 - \langle z, \eta \rangle)^N} \cdot \frac{(1 - |a|^2)^N}{|1 - \langle \eta, a \rangle|^{2N}} d\sigma(\eta) \\
 &= (1 - \langle z, a \rangle)^N \int_{\partial B_N} f \circ \varphi(\eta) g \circ \varphi(\eta) \frac{1}{(1 - \langle \eta, a \rangle)^N} \cdot \frac{1}{(1 - \langle z, \eta \rangle)^N} d\sigma(\eta) \\
 &= (1 - \langle z, a \rangle)^N (T_{f \circ \varphi, K_a} g \circ \varphi)(z) \\
 &= (T_{1/K_a} T_{f \circ \varphi} T_{K_a} C_\varphi g)(z),
 \end{aligned}$$

where we have used the change of variables formula (see Corollary 4.4 of [20]) and the kernel function

$$K_a(z) = \frac{1}{(1 - \langle z, a \rangle)^N}$$

and the function $1/K_a$ are analytic on $\overline{B_N}$. Hence,

$$C_\varphi T_f = T_{1/K_a} T_{f \circ \varphi} T_{K_a} C_\varphi.$$

Since f is continuous on ∂B_N , by the semi-multiplicative property for Toeplitz operators mod \mathcal{K} in Section 2, we know that

$$T_{1/K_a} T_{f \circ \varphi} = T_{1/K_a \cdot f \circ \varphi} + L = T_{f \circ \varphi, 1/K_a} + L = T_{f \circ \varphi} T_{1/K_a} + L,$$

where L is a compact operator on $H^2(B_N)$. Therefore,

$$C_\varphi T_f = T_{1/K_a} T_{f \circ \varphi} T_{K_a} C_\varphi = (T_{f \circ \varphi} T_{1/K_a} + L) T_{K_a} C_\varphi = T_{f \circ \varphi} C_\varphi + L',$$

where L' is compact. Applying similar technique, we can obtain the similar result for the weighted Bergman space and so we complete the proof. \square

Based on these lemmas, we will use another technique to prove Theorem 3.1, which generalizes Theorem 5.2 of [4] and Theorem 5.1 of [15] to the unit ball.

Proof of Theorem 3.1. We only need to prove the "only if" part and we only give a proof for the Hardy space case.

Assume that $[C_\psi^*, C_\varphi]$ is compact. Since ψ is an automorphism of B_N , by Lemma 3.6, we have

$$C_\psi^* = T_f C_{\psi^{-1}}$$

with

$$f(z) = \left(\frac{1 - |\psi(0)|^2}{|1 - \langle z, \psi(0) \rangle|^2} \right)^N.$$

Thus,

$$[C_\psi^*, C_\varphi] = C_\psi^* C_\varphi - C_\varphi C_\psi^* = T_f C_{\psi^{-1}} C_\varphi - C_\varphi T_f C_{\psi^{-1}}.$$

Now, using Lemma 3.5, $[C_\psi^*, C_\varphi]$ is compact means that φ and ψ commute, i.e.

$$\varphi \circ \psi = \psi \circ \varphi.$$

This gives $\varphi = \psi \circ \varphi \circ \psi^{-1}$ and

$$\begin{aligned} [C_\psi^*, C_\varphi]C_\psi &= (T_f C_{\psi^{-1}} C_\varphi - C_\varphi T_f C_{\psi^{-1}})C_\psi \\ &= T_f C_{\psi^{-1}} C_\varphi C_\psi - C_\varphi T_f C_{\psi^{-1}} C_\psi \\ &= T_f C_{\psi \circ \varphi \circ \psi^{-1}} - C_\varphi T_f \\ &= T_f C_\varphi - C_\varphi T_f. \end{aligned}$$

Note that f is continuous on $\overline{B_N}$ and φ is an automorphism of B_N . It follows from Lemma 3.7 that

$$C_\varphi T_f \equiv T_{f \circ \varphi} C_\varphi \pmod{\mathcal{K}}.$$

Therefore,

$$\begin{aligned} [C_\psi^*, C_\varphi]C_\psi &= T_f C_\varphi - C_\varphi T_f \\ &\equiv T_f C_\varphi - T_{f \circ \varphi} C_\varphi \pmod{\mathcal{K}} \\ &= T_{f - f \circ \varphi} C_\varphi \pmod{\mathcal{K}}. \end{aligned}$$

Finally, since C_ψ and C_φ are invertible, we see that $[C_\psi^*, C_\varphi]$ is compact if and only if $[C_\psi^*, C_\varphi]C_\psi$ is compact, which is equivalent to that $T_{f - f \circ \varphi}$ is compact. Applying Lemma 2 of [5], we see that $f - f \circ \varphi \equiv 0$ on $\overline{B_N}$. Since φ is not the identity, the representation of f gives that f must be constant. Thus, we get $\psi(0) = 0$.

As we know, an automorphism ϕ of B_N is a unitary transformation of \mathbb{C}^N if and only if $\phi(0) = 0$ (see Lemma 1.1 of [20]). Together this with the above discussion, we see that ψ must be unitary. On the other hand, $[C_\psi^*, C_\varphi] = C_\psi^* C_\varphi - C_\varphi C_\psi^*$ is compact implies that

$$(C_\psi^* C_\varphi - C_\varphi C_\psi^*)^* = C_\varphi^* C_\psi - C_\psi C_\varphi^* = [C_\varphi^*, C_\psi]$$

is compact. So similar arguments deduce that φ is also unitary. Consequently, all these give that both φ and ψ are unitary and they commute under the condition that $[C_\psi^*, C_\varphi]$ is compact. \square

4. THE COMMUTATOR ON THE DIRICHLET SPACE

In this section, we try to characterize the compactness of $[C_\psi^*, C_\varphi]$ on the Dirichlet space $\mathcal{D}(B_N)$, where φ and ψ are linear fractional self-maps of B_N . The following lemma is about compact difference of linear fractional composition operators on $\mathcal{D}(B_N)$, which will prove useful for our result.

Lemma 4.1. *Suppose that φ and ψ are linear fractional self-maps of B_N . Then $C_\varphi - C_\psi$ is compact on $\mathcal{D}(B_N)$ if and only if $\varphi = \psi$ or both C_φ and C_ψ are compact.*

Proof. This result for a special case has been pointed out by Pons [18], without proof. For completeness, we give a simple proof.

In [18], for real s , the weighted Dirichlet space $\mathcal{D}_s(B_N)$ is defined by

$$\mathcal{D}_s(B_N) = \{f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha} \text{ analytic in } B_N : \sum_{\alpha} (|\alpha| + 1)^{1-s} |c_{\alpha}|^2 \omega_{\alpha} < \infty\},$$

where

$$\omega_{\alpha} = \|z^{\alpha}\|^2 = \frac{(N-1)!\alpha!}{(N-1+|\alpha|)!}.$$

On the other hand, let $\beta(k) = (k+1)^t$ for real t , if $f(z) = \sum c_\alpha z^\alpha = \sum_0^\infty f_k(z)$ is analytic in B_N , then f belongs to the weighted Hardy space $H^2(\beta, B_N)$ if and only if

$$\sum_0^\infty \|f_k\|^2 \beta(k)^2 = \sum_\alpha (|\alpha| + 1)^{2t} |c_\alpha|^2 \omega_\alpha < \infty.$$

Thus, the weighted Dirichlet space $\mathcal{D}_s(B_N)$, in fact, is the weighted Hardy space $H^2(\beta, B_N)$ with the weight $\beta(k) = (k+1)^{(1-s)/2}$. Using Proposition 2.4 of [13], we see that all linear fractional self-maps induce bounded composition operators on $\mathcal{D}_s(B_N)$.

Suppose $s_1 < s < s_2 < \infty$, complex interpolation theorem for the weighted Dirichlet space $\mathcal{D}_s(B_N)$ tells us

$$[\mathcal{D}_{s_1}, \mathcal{D}_{s_2}]_\theta = \mathcal{D}_s$$

with $s = (1-\theta)s_1 + \theta s_2$ for $\theta \in (0, 1)$ (see Proposition 1 of [18]). Choosing $s_1 = -n$ and $s_2 = 2$, then for linear fractional self-maps φ and ψ of B_N , the operator $C_\varphi - C_\psi$ is bounded on $\mathcal{D}_{-n}(B_N)$ and $\mathcal{D}_2(B_N) = A^2(B_N)$. Now, if $C_\varphi - C_\psi$ is compact on $\mathcal{D}(B_N) = \mathcal{D}_{1-n}(B_N)$. Since $1-n = (1-\theta)(-n) + \theta \cdot 2$ with $\theta = \frac{1}{n+2}$, using the compactness theorem for interpolating operators (see Theorem 2.1 in [9]), then $C_\varphi - C_\psi$ is also compact on $H^2(B_N) = \mathcal{D}_1(B_N)$ with $1 = (1-\theta)(-n) + \theta \cdot 2$ for $\theta = \frac{n+1}{n+2}$. Here, all above spaces are identified to equal with an equivalent norm. Applying Theorem 2 of [11] or Theorem 3.1 of [14], we get that $\varphi = \psi$ or both C_φ and C_ψ are compact. So we have shown one direction. Another direction is obvious and the proof is completed. \square

When φ is a linear fractional self-map of B_N , the adjoint of composition operator C_φ on $\mathcal{D}(B_N)$ is mainly determined by another composition operator. Together this adjoint property with Lemma 4.1, we will give the following condition for $[C_\psi^*, C_\varphi]$ to be compact on $\mathcal{D}(B_N)$.

Theorem 4.2. *Let φ and ψ be linear fractional self-maps of B_N . If $[C_\psi^*, C_\varphi] \neq 0$, then $[C_\psi^*, C_\varphi]$ is non-trivially compact on $\mathcal{D}(B_N)$ if and only if $\|\psi\|_\infty = \|\varphi\|_\infty = 1$ and $\varphi \circ \sigma = \sigma \circ \varphi$, where σ is the Krein adjoint of ψ .*

Proof. For $w \in B_N$, let K_w denote the reproducing kernel of $\mathcal{D}(B_N)$. Theorem C gives

$$C_\psi^* f = f(0)K_{\psi(0)} + C_\sigma f - f(\sigma(0))$$

for any $f \in \mathcal{D}(B_N)$, where σ is the Krein adjoint of ψ . Thus,

$$\begin{aligned} [C_\psi^*, C_\varphi]f &= (C_\psi^* C_\varphi - C_\varphi C_\psi^*)f = C_\psi^* C_\varphi f - C_\varphi C_\psi^* f \\ &= f \circ \varphi(0)K_{\psi(0)} - C_\sigma C_\varphi f - f \circ \varphi \circ \sigma(0) \\ &\quad - [f(0)K_{\psi(0)} \circ \varphi - C_\varphi C_\sigma f - f(\sigma(0))] \\ &= (C_\varphi C_\sigma - C_\sigma C_\varphi)f + f \circ \varphi(0)K_{\psi(0)} + f(\sigma(0)) \\ &\quad - f \circ \varphi \circ \sigma(0) - f(0)K_{\psi(0)} \circ \varphi. \end{aligned}$$

This implies that $[C_\psi^*, C_\varphi]$ is compact on $\mathcal{D}(B_N)$ if and only if

$$[C_\varphi, C_\sigma] = C_\varphi C_\sigma - C_\sigma C_\varphi = C_{\sigma \circ \varphi} - C_{\varphi \circ \sigma}$$

is compact.

It is easy to see that $C_\psi^* C_\varphi$ is compact if and only if $C_\sigma C_\varphi = C_{\varphi \circ \sigma}$ is compact, and the compactness of $C_\varphi C_\psi^*$ is equivalent to the compactness of $C_\varphi C_\sigma = C_{\sigma \circ \varphi}$. Since $[C_\psi^*, C_\varphi] \neq 0$, we get that $[C_\psi^*, C_\varphi]$ is non-trivially compact on $\mathcal{D}(B_N)$ if and only

if $C_{\sigma \circ \varphi} - C_{\varphi \circ \sigma}$ is also non-trivially compact. By Lemma 4.1, this is equivalent to $\varphi \circ \sigma = \sigma \circ \varphi$ and $\|\varphi \circ \sigma\|_\infty = \|\sigma \circ \varphi\|_\infty = 1$, i.e. $\|\psi\|_\infty = \|\varphi\|_\infty = 1$. So we complete the proof. \square

Note that if ψ is an automorphism of B_N , then $\sigma = \psi^{-1}$. Thus, $\varphi \circ \sigma = \sigma \circ \varphi$ gives that $\varphi \circ \psi^{-1} = \psi^{-1} \circ \varphi$. This is the same to $\varphi \circ \psi = \psi \circ \varphi$. Immediately, as a corollary of Theorem 4.2, we obtain the following result.

Theorem 4.3. *If φ and ψ are automorphisms of B_N . Then $[C_\psi^*, C_\varphi]$ is compact on $\mathcal{D}(B_N)$ if and only if φ and ψ commute.*

Using Theorem 4.2 and similar discussions in the proof of Theorem 3.1 in [15], we also obtain the following similar result on the Dirichlet space $\mathcal{D}(D)$ as on the Hardy space $H^2(D)$ and the weighted Bergman space $A_s^2(D)$ ($s > -1$).

Theorem 4.4. *Suppose that φ and ψ are linear fractional self-maps of D , and one of which is a non-automorphism. The commutator $[C_\psi^*, C_\varphi]$ is non-trivially compact on $\mathcal{D}(D)$ if and only if one of the following is true.*

- (i) φ and ψ are both parabolic with the same boundary fixed point,
- (ii) φ and ψ are hyperbolic such that the fixed points of φ and ψ are (ζ, a) and $(\zeta, 1/\bar{a})$ respectively with $\zeta \in \partial D$.

Remark. We first see the following example. Let

$$\varphi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1, z_2/2), \quad z = (z_1, z_2) \in B_2$$

and

$$\psi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1, z_2/3), \quad z = (z_1, z_2) \in B_2.$$

Thus,

$$\sigma(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and so

$$\varphi \circ \sigma(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \sigma \circ \varphi(z).$$

Moreover, it is easy to check that

$$\begin{aligned} [C_\psi^*, C_\varphi]f &= (C_\varphi C_\sigma - C_\sigma C_\varphi)f + f \circ \varphi(0)K_{\psi(0)} + f(\sigma(0)) \\ &\quad - f \circ \varphi \circ \sigma(0) - f(0)K_{\psi(0)} \circ \varphi \\ &= 0 \end{aligned}$$

for any $f \in \mathcal{D}(B_2)$. As a consequence, when $\varphi \circ \sigma = \sigma \circ \varphi$ and $\|\psi\|_\infty = \|\varphi\|_\infty = 1$, it may happen that $[C_\psi^*, C_\varphi] = 0$. Hence, in Theorem 4.2, we need assume that $[C_\psi^*, C_\varphi] \neq 0$. However, from the proof of Theorem 3.1 in [15], this condition is not necessary for the unit disk.

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